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ON DIFFERENCE APPROXIMATIONS OF OPTIMAL CONTROL SYSTEMS

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The approximation of continuous-time optimal control problems by sequences of finite-dimensional (discrete-time) optimization problems, arising from difference replacement of derivatives, is investigated. Necessary and sufficient condition for the convergence of discrete (finite difference) approximations with respect to a functional is obtained under minimal assumptions and estimates of convergence rate are found. The results obtained permit the justification of numerical methods for solving optimal control problems on a computer and the investigation of a number of interrelated qualitative aspects of the optimization of continuous and discrete control systems.

1. Statement of the problem. We consider the following optimal control problem for systems of ordinary differential equations

$$x^{\cdot} = f(x, u, t), \quad x(t_0) = x_0 \tag{1.1}$$

$$u(t) \in U, \quad t \in T = [t_0, t_1] \tag{1.2}$$

$$x(t) \in G(t) \subset \mathbf{R}^n, \quad t \in T \tag{1.3}$$

$$I = \varphi(x(t_1)) \to \inf \tag{1.4}$$

The vector $x_0 \in \mathbf{R}^n$ in (1.1) and the instants t_0 and t_1 are taken as fixed. The solution of problem (1.1) - (1.4) (for convenience we call it Problem A) is sought in the class of measurable controls $u(t)$ and of absolutely continuous trajectories $x(t)$, where it is assumed that at least one admissible pair $\{x(t), u(t)\}$, $t \in T$ exists. We remark that a number of other optimization problems for systems of ordinary differential equations with fixed and nonfixed time in the presence of phase and integral constraints reduce to Problem A.

We approximate Problem A by a sequence of discrete-approximation Problems A_N obtained from A by a difference replacement of derivative $x^{\cdot}(t)$ at specified points of partitioning of interval T. For each positive integer N we consider a difference partitioning $T_N = \{t_0, t_0 + h_{1N}, \ldots, t_1\}$ of interval T with variable step h_{kN}, $k = 1, \ldots, m(N)$, and a set of $G_N(t) \subset \mathbf{R}^n$, $t \in T$ approximating constraint (1.3). Problem A_N consists in the minimization of functional (1.4) under the following constraints:

$$\begin{gathered} x_N(t + h_{kN}) = x_N(t) + h_{kN} f(x_N(t), u_N(t), t) \\ x_N(t_0) = x_0, \quad t = t_0 + \sum_{i=1}^{k-1} h_{iN}, \quad k = 1, \ldots, m(N) \end{gathered} \tag{1.5}$$

$$u_N(t) \in U, \quad t \in T_N \setminus t_1 = T_N{}^1 \tag{1.6}$$

$$x_N(t) \in G_N(t), \quad t \in T_N \tag{1.7}$$

We introduce the notation

$$h_N = \max_{1 \leqslant k \leqslant m(N)} h_{kN}, \quad \beta_N = \sup_{t \in T} \sup_{x \in G_N(t)} \inf_{z \in G(t)} \| x - z \|$$

where $\| \cdot \|$ is some norm in space $\mathbf{R}^n$. The problem sequence $\{A_N\}$, $N = 1, 2, \ldots$ is called a discrete approximation of Problem A if $h_N \to 0$ and $\beta_N \to 0$ as $N \to \infty$. The set sequence $\{G_N(t)\}$, $N = 1, 2, \ldots$, is called a ρ_N approximation of set $G(t)$ if

$$G_N(t) \supset [G(t)]_{\rho_N} \overset{\text{def}}{=} \{x \in \mathbf{R}^n: \inf_{z \in G(t)} \| x - z \| \leqslant \rho_N\}, \quad N = 1, 2, \ldots \tag{1.8}$$

for all $t \in T$.

The present paper's purpose is to find the conditions under which the optimal values of the functional being minimized in Problems A_N with large N are as close as required to the optimal value of functional (1.4) in Problem A (the convergence $A_N \to A$ with respect to the functional) holds. On this we base a direct method of solving optimal control problems with phase constraints connected with difference replacements of derivatives and with passing to finite-dimensional discrete optimization problems.

The first similar investigations for a linear time-optimality problem were conducted by Krasovskii [1]. Subsequent results in this direction are presented in [2–8] and others wherein a number of sufficient conditions are obtained for the convergence of discrete approximations with respect to a functional for problems of the type being analyzed, under various methods of approximating constraints (1.2) and (1.3). In the present paper we develop a new approach to the investigation of discrete approximations of general optimal control problems and we have obtained a necessary and sufficient condition for convergence $A_N \to A$ with respect to the functional together with estimates of convergence rate. The methods of the theory of existence of optimal controls [9] are used to prove the theorems. The results obtained find application in the construction of the approximations and in the proof by the scheme in [10] of the maximum principle in nonsmooth optimal control problems with phase constraints.

2. Correct formulation with respect to extension. In what follows we assume the fulfilment of the following general conditions on the parameters of the problems being analyzed:

a) the control domain U is a metric compactum;

b) the sets $G(t)$ and $G_N(t)$ are closed in $\mathbf{R}^n$ for all $t \in T$, and $G(t)$ is upper-semicontinuous at all points $t \in (t_0, t_1)$ not common for the sequence of partitionings $T_N{}^1$, $N \geqslant N_0$;

c) the admissible trajectories of Problems A and A_N, $N \geqslant N_0$, do not go outside a certain sphere $S_r = \{x \in R^n : \| x \| \leqslant r\}$, $0 < r < \infty$, (sufficient conditions for this are given in [9]);

d) the functions $f(x, u, t)$ and $\varphi(x)$ are continuous on sets $Z = S_r \times U \times T$ and S_r, respectively;

e) the Cauchy problem (1.1), (1.2) has a unique solution.

Together with the original Problem A we consider an auxiliary optimal control Problem B which is the Gamkrelidze - extension [9, 11] of Problem A: minimize functional (1.4) on the set of measurable controls $\{\alpha_i(t), u_i(t), \quad i = 1, \ldots, n+1\}$ and of absolutely continuous trajectories $x(t), \ t_0 \leqslant t \leqslant t_1$, satisfying constraints (1.3) in the following relations:

$$x^{\cdot} = \sum_{i=1}^{n+1} \alpha_i f(x, u_i, t), \quad x(t_0) = x_0 \tag{2.1}$$

$$\alpha_i(t) \geqslant 0, \ \sum_{i=1}^{n+1} \alpha_i(t) = 1, \quad u_i(t) \in U(t) \tag{2.2}$$

$$t \in T, \ i = 1, \ldots, n+1$$

By I_A°, I_B° and I_N°, $N = 1, 2, \ldots$, we denote the minimal values of functional (1.4) in Problems A, B and A_N respectively. Following [12], we say that Problem A is correctly formulated with respect to extension if $I_A^\circ = I_B^\circ$. Correct formulation with respect to extension is a natural property of control systems which is violated, as a rule, only in special " poorly formulated" optimization problems.

A number of general conditions for correct formulation were obtained in [13, 14] and others (see [9], wherein broad classes of optimal control problems, correctly formulated with respect to extension, with a nonconvex set of admissible velocities have been distinguished. In particular, Problems A without phase constraints (1.3), with the right-hand side linear with respect to the state or one-dimensional, normal in the sense of the maximum principle, etc., are correctly formulated. It is shown below that correct formulation with respect to extension is a necessary and sufficient condition for the convergence of discrete approximations with respect to the functional.

3. Approximations of continuous curves. Let us prove the possibility of a uniform approximation of admissible trajectories of Problem A by a sequence of corresponding trajectories of discrete systems. Let $\{x_N^{\cdot}(t), u_N(t)\}$ be a discrete pair satisfying (1.5) and (1.6). For an arbitrary point $t \in T$ we denote the elements of partitioning T_N., closest to the left and to the right of it, by t^N and t_N and consider the piecewise-linear continuation of trajectory $x_N(t)$ onto the whole interval T

$$x_N(t) = x_N(t^N) + \frac{1}{t_N - t^N}[x_N(t_N) - x_N(t^N)](t - t^N), \quad t_0 \leqslant t \leqslant t_1 \tag{3.1}$$

Theorem 3.1. Let conditions a) and c) — e) be fulfilled. Then for any trajectory $x(t)$ admissible in (1.1) — (1.3) and for any choice of a sequence of partitionings $\{T_N\}, N = 1, 2, \ldots$, of interval T we can find a sub-sequence of discrete pairs $\{x_N(t), u_N(t)\}$, $N \to \infty$ and $N \in \Lambda$, for which relations (1.5) and (1.6) are fulfilled and continuation (3.1) converges to $x(t)$ uniformly on T.

Proof. We first consider the case when control $u(t)$ corresponding by virtue of (1.1) and (1.2) to the selected trajectory $x(t)$ is continuous at almost all points of interval T. From the function $u(t)$ specified we form the discrete controls $u_N(t) = u(t), \ t \in T_N^1, \ N = 1, 2, \ldots$, and we show that the corresponding sequence of trajectories $x_N(t)$ of system (1.5) converges to $x(t)$ uniformly on T. By virtue of (1.5) and (3.1) we have

$$x_N^{\cdot}(t) = f(x_N(t^N), \quad u_N(t^N), t^N), \qquad t \in T \setminus T_N \tag{3.2}$$

From the theorem 's hypotheses we conclude that the sequence $\{x_N(t)\}$, $t \in T$, $N = 1, 2, \ldots$, is uniformly bounded and equicontinuous; consequently, it contains a uniformly convergent subsequence. By $x^*(t)$ we denote any limit point in $C(T)$ of the sequence $\{x_N(t)\}$, $N = 1, 2, \ldots$, and we prove that $x^*(t) \equiv x(t)$. We consider the functions $h_N(t) = f(x_N(t^N), u_N(t^N), t^N)$ on the whole interval T. From the continuity almost everywhere on T of control $u(t)$ and from the construction of the discrete controls $u_N(t)$ it follows that the sequence $\{h_N(t)\}$ converges almost everywhere to the function $f(x^*(t), u(t), t)$. Passing to the limit in

$$x_N(t) = x_0 + \int_{t_0}^{t} h_N(\tau)\, d\tau, \quad t_0 \leqslant t \leqslant t_1$$

and making use of Lebesgue's theorem on passing to the limit under the integral sign [15], we find that function $x^*(t)$ is the solution of system (1.1) with $u = u\cdot(t)$. The required equality $x^*(t) \equiv x(t)$ now follows from the uniqueness of the solution of the Cauchy problem for (1.1) with $u = u(t)$; this proves the theorem when control $u(t)$ is continuous almost everywhere.

The general case of a measurable control $u(t)$ is reduced to the one already considered by using the following statement. Any measurable function $u(t)$ satisfying constraint (1.2) can be approximated in the sense of convergence in measure by a sequence of functions, continuous almost everywhere on T, with the same constraint. In this connection the convergence of the corresponding trajectories is uniform.

To prove this statement we make use of Luzin's theorem (the C-property of measurable functions) [15]. We consider an arbitrary sequence of positive numbers $\varepsilon_k \to 0$, $k = 1, 2, \ldots$, and we find closed sets T_{ε_k} having the properties that $\mathrm{mes}(T \setminus T_{\varepsilon_k}) \leqslant \varepsilon_k$ and that the restriction of function $u(t)$ on T_{ε_k} is continuous. The set $T \setminus T_{\varepsilon_k}$ can be given as the union of a denumerable number of nonintersecting intervals $(\alpha_{jk}, \beta_{jk})$, $j = 1, 2, \ldots$ Let us consider the functions

$$u_k(t) = \begin{cases} u(t), & t \in T_{\varepsilon_k}, \qquad k = 1, 2, \ldots \\ u(\alpha_{jk}), & t \in (\alpha_{jk}, \beta_{jk}), \quad j = 1, 2 \ldots \end{cases} \tag{3.3}$$

From the forms of (3.3) we conclude that functions $u_k(t)$ can be discontinuous only at the points $t = \beta_{jk}$, $j = 1, 2, \ldots$. In addition $\mathrm{mes}\,\{t : u_k(t) \neq u(t)\} \leqslant \varepsilon_k$, which ensures the convergence $u_k(t) \to u(t)$ in measure. Thus, the sequence $\{u_k(t)\}$, $k = 1, 2, \ldots$, is the one desired. The uniform convergence of the corresponding trajectories follows from Lebesgue's theorem [15].

4. Convergence with respect to the functional. Let us state and prove the main result, viz., a theorem on the convergence of discrete approximations with respect to the functional.

Theorem 4.1. Let conditions a) — e) be fulfilled. Then for the sequence of partitionings $\{T_N\}$ of interval T we can find a numerical sequence $\{\rho_N\}$, $N \to \infty$, $N \in \Lambda$, $\rho_N \to 0$, for which the inequalities

$$I_B^\circ \leqslant \liminf_{N\to\infty\; N\in\Lambda} I_N^\circ \leqslant \limsup_{N\to\infty\; N\in\Lambda} I_N^\circ \leqslant I_A^\circ \tag{4.1}$$

hold for any ρ_N- approximation $\{G_N(t)\}$ of set $G(t)$, which guarantee the convergence $A_N \to A$ with respect to the functional under the condition that Problem A is correctly formulated with respect to extension. If function $f(x, u, t)$ satisfies the Lipschitz condition in x on set $Z = S_r \times U \times T$, then the converse statement is valid: the correct formulation with respect to extension of Problem A follows from the convergence $A_N \to A$ with respect to the functional under any ρ_N-approximation of set $G(t)$.

P r o o f. Let $\{x_k(t)\}$, $t_0 \leqslant t \leqslant t_1$, $k = 1, 2, \ldots$, be a minimizing sequence of admissible trajectories in Problem A. From the theorem's hypotheses we conclude that sequence $\{x_k(t)\}$ is relatively compact in space $C(T)$. Consequently, we can find a subsequence from $\{x_k(t)\}$, $k = 1, 2, \ldots$, converging uniformly on T to the absolutely continuous function $x^\circ(t)$ minimizing functional (1.4) in Problem A. Let $\{T_N\}$, $N = 1, 2, \ldots$, be any sequence of partitionings of interval T with the maximal partitioning step $h_N \to 0$. Using Theorem 3.1 we can select a subsequence of discrete trajectories $x_N(t)$, $N \to \infty$ and $N \in \Lambda$, admissible in (1.5) and (1.6), whose continuous extensions converge uniformly on T to the indicated limit $x^\circ(t)$.

We consider an arbitrary sequence of discrete Problems A_N, in which the sets $G_N(t)$ form a ρ_N- approximation of set $G(t)$ and the numbers ρ_N are chosen from the condition

$$\rho_N \geqslant \max_{t_0\leqslant t\leqslant t_1} \|x^\circ(t) - x_N(t)\|, \quad \rho_N \to 0, \quad N \to \infty, \quad N \in \Lambda \tag{4.2}$$

Let us prove that relations (4.1) are fulfilled for the sequence of Problems A_N being examined. At first we prove the validity of the inequality on the right in (4.1). We assume that it does not hold, i.e., the inequality

$$\lim_{N\to\infty,\, N\in\Lambda_1} \overset{\circ}{I_N} > I_A^\circ = \varphi(x^\circ(t_1))$$

is fulfilled for some subsequence $\Lambda_1 = \{N\} \subset \Lambda$. Then for sufficiently large $N \in \Lambda_1$ we have

$$I_N^\circ > \varphi(x_N(t_1)) \tag{4.3}$$

From the construction of Problems A_N it follows that the trajectories $x_N(t)$ are admissible in them since the inclusions

$$x_N(t) \in [G(t)]_{\rho_N} \subset G_N(t), \quad t \in T_N$$

are valid by virtue of (1.3), (1.8) and (4.2). Consequently, (4.3) cannot hold, i.e., the inequality on the right in (4.1) is fulfilled.

To prove the inequality on the left in (4.1) it suffices to show that the uniform limit of the sequence of trajectories $x_N(t)$, $t \in T$, admissible in Problems A_N is an admissible trajectory of the extended Problem B. From condition b) it follows that the limit function $x(t)$ satisfies constraint (1.3). Let us prove that the inclusion

$$x^{\cdot}(t) \in R(x(t), t) \overset{\text{def}}{=} \operatorname{conv} f(x(t), U, t) \tag{4.4}$$
$$f(x, U, t) = \{v \in \mathbf{R}^n : v = f(x, u, t),\ u \in U\}$$

is valid for almost all $t \in T$. Here $f(x, U, t)$ is the set of velocities admissible in (1.1) and (1.2) and $\operatorname{conv} V$ denotes the convex hull of set V. Using the theorem on measurable selectors [9] we can deduce from (4.4) that the trajectory $x(t)$ in (2.1) and (2.2) is realized by the measurable control $\{\alpha_i(t), u_i(t), i = 1, \ldots, n+1\}$, i.e., is admissible in Problem B.

To prove (4.4) we take any $\varepsilon > 0$ and we write the inclusion

$$x_N^{\cdot}(t) = f(x_N(t^N), \quad u_N(t^N), t^N) \in [f(x(t), U, t)]_\varepsilon$$

valid for all $N \geqslant N_0$ and for almost all $t \in T$. The theorem's hypotheses let's us conclude that the sequence $\{x_N^{\cdot}(t)\}$, $N \geqslant N_0$, converges to $x^{\cdot}(t)$ weakly in $L_2(T)$. Applying Mazur's weak closure theorem [15] to the sequence, we find that $x^{\cdot}(t) \in \operatorname{conv}[f(x(t), U, t)]_\varepsilon$ for any $\varepsilon > 0$, i.e., the inclusion

$$x^{\cdot}(t) \in \bigcap_{\varepsilon > 0} \operatorname{conv}[f(x(t), U, t)]_\varepsilon$$

is valid for almost all $t \in T$. To prove (4.4) it is enough to show that the equality

$$R(x, t) = \bigcap_{\varepsilon > 0} \operatorname{conv}[f(x, U, t)]_\varepsilon \tag{4.5}$$

holds under the theorem's hypotheses. Obviously, the left-hand side of (4.5) is contained in the right-hand one. Let us prove the reverse inclusion. Let v be an element of the set on the right-hand side of (4.5). Then, for any numerical sequence $\varepsilon_k \downarrow 0$, $k = 1, 2, \ldots$, we can find a sequence of vectors $\{\alpha_i^k, v_i^k\}$, $i = 1, \ldots, n+1$ and $k = 1, 2, \ldots$, for which

$$v = \sum_{i=1}^{n+1} \alpha_i^k v_i^k, \quad v_i^k \in [f(x, U, t)]_{\varepsilon_k}, \quad \alpha_i^k \geqslant 0, \sum_{i=1}^{n+1} \alpha_i^k = 1 \tag{4.6}$$

By u_i^k we denote the points of set U, for which

$$\| f(x, u_i^k, t) - v_i^k \| \leqslant \varepsilon_k, \quad i = 1, \ldots, n+1;\ k = 1, 2, \ldots \tag{4.7}$$

Using the compactness of sets U, $f(x, U, t)$ and $P = \{\alpha_i : \alpha_i \geqslant 0, \alpha_1 + \alpha_2 + \ldots + \alpha_{n+1} = 1\}$, we pick out a subset of indices $\{k\}$, $k \to \infty$, for which

$$u_i^k \to u_i^\circ \in U, \quad \{\alpha_i^k\} \to \{\alpha_i^\circ\} \in P, \quad v_i^k \to v_i^\circ, \quad i = 1, \ldots, n+1$$

Because of (4.6) and (4.7) we have

$$v_i^\circ = f(x, u_i^\circ, t), \quad i = 1, \ldots, n+1, \quad v = \sum_{i=1}^{n+1} \alpha_i^\circ f(x, u_i^\circ, t)$$

which proves inclusions (4.4) and (4.5) and the inequality on the left in (4.1). Thus, we have proved that inequalities (4.1) hold for any ρ_N- approximation of set $G(t)$ when the sequence $\{\rho_N\}$ is chosen in accordance with (4.2) and the correct formulation of Problem A with respect to extension is a sufficient condition for the convergence $A_N \to A$ with respect to the functional.

Let us now prove that the condition of correct formulation of Problem A with respect to extension is also necessary for the convergence $A_N \to A$ with respect to the functional for any ρ_N- approximation of set $G(t)$. We assume that Problem A is not correctly formulated with respect to extension, i.e., $I_B^\circ < I_A^\circ$. Under the theorem's hypotheses, an optimal trajectory $x^\circ(t)$ exists in Problem B, which can be approximated uniformly on T by a sequence of absolutely continuous functions $x_k(t)$ satisfying together with certain measurable controls $u_k(t)$ the constraints (2.1) and (2.2), $k = 1, 2, \ldots$ [9,11]. Using Theorem 3.1 we form the sequence of discrete trajectories $\{x_N(t)\}$, $N \in \Lambda$ and $N \to \infty$, whose continuations (3.1) converge to $x^\circ(t)$ uniformly on T. We choose numbers ρ_N from (4.2) and we consider the corresponding sequence of Problems A_N forming the ρ_N- approximation of Problem A. Using the preceding arguments, we arrive at the relation

$$\lim_{N\to\infty,\, N\in\Lambda} I_N^\circ = I_B^\circ < I_A^\circ$$

by virtue of which the Problems A_N being examined do not converge to A with respect to the functional.

Notes. 4.1. The Lipschitz condition in the proof of necessity in Theorem 4.1 was required in order that the trajectories admissible in Problem B could be approximated by admissible trajectories in (1.1), (1.2). The latter assertion is valid under more general assumptions ensuring the uniqueness of the solution of the Cauchy problem in (2.1), (2.2).

4.2. An analogous theorem on the convergence of discrete approximations with respect to the functional is valid under other (more precise) methods of difference approximation of the derivative.

5. Estimates of the approximation and of the convergence rate. Let us find the conditions under which we can effectively estimate the quantities ρ_N in Theorem 4.1 and the rate of convergence $A_N \to A$ with respect to the functional. By $\rho_U(x, y)$ we denote the distance function in space U.

Theorem 5.1. In addition to conditions a) – e) we assume that function $f(x, u, t)$ satisfies on Z a Lipschitz condition in all variables with a constant L_f, and that an optimal control $u^\circ(t)$, continuous almost everywhere, exists in Problem A, with the following property: for a specified sequence of partitionings $\{T_N\}$ of interval T we can find a nondecreasing function $\omega(\eta)$, $0 \leqslant \eta < \infty$, $\lim_{\eta\to 0} \omega(\eta) = \omega(0) = 0$, and a number N_0 for which

$$\rho_U(u^\circ(t),\ u^\circ(t^N)) \leqslant \omega(t - t^N), \quad N \geqslant N_0 \tag{5.1}$$

is valid for almost all $t \in T$. Then as the $\{\rho_N\}$, $N = 1,2, \ldots$, in Theorem 4.1 we can take any sequence of positive numbers satisfying the condition

$$\rho_N \geqslant M_1 h_N + M_2 \omega(h_N), \quad \rho_N \to 0, \quad N \to \infty \tag{5.2}$$

$$(M_1 = {}^1/_2 L_f T(\mu + 1)\exp[L_f T], \ M_2 = L_f T \exp[L_f T]$$

$$\mu = \max_{(x,u,t)\in Z} \|f(x, u, t)\|)$$

If function $\varphi(x)$ satisfies a Lipschitz condition on S_r with constant L_φ, then the upper bound

$$-L_\varphi M_1 h_N \leqslant I_N^\circ - I_A^\circ \leqslant L_\varphi M_1 h_N + L_\varphi M_2 \omega(h_N) \tag{5.3}$$

is valid for the rate of convergence $A_N \to A$ with respect to the functional. The lower bound in (5.3) is automatically fulfilled under conditions a), c), $G(t) \equiv \mathbf{R}^n$ and the validity of the Lipschitz conditions on functions φ and f in (x, t).

Proof. From the constructions in Theorems 3.1 and 4.1 it follows that under the assumptions made the desired sequences $\{\rho_N\}$ are chosen from condition (4.2) wherein as $x^\circ(t)$ we can take the trajectory of (1.1), corresponding to the optimal control $u^\circ(t)$, being examined, and as $x_N(t)$ we can take the solutions of system (1.5), corresponding to the discrete controls $u_N(t) = u^\circ(t)$, $t \in T_N^1$, $N = 1,2 \ldots$

Let us estimate the difference $\Delta(t) = \| x^\circ(t) - x_N^*(t) \|$. By virtue of (3.2), (5.1) and the theorem's hypotheses we have

$$\begin{aligned}
\Delta(t) &\leqslant \int_{t_0}^{t} \| f(x^\circ(\tau), u^\circ(\tau), \tau) - f(x_N(\tau^N), u_N^\circ(\tau^N), \tau^N) \| \, d\tau \leqslant \\
& L_f \int_{t_0}^{t} \Delta(\tau)\, d\tau + L_f \int_{t_0}^{t} \rho_U(u^\circ(\tau), u^\circ(\tau^N))\, d\tau + \\
& L_f(\mu+1) \int_{t_0}^{t} (\tau - \tau^N)\, d\tau \leqslant L_f \int_{t_0}^{t} \Delta(\tau)\, d\tau + L_f T \omega(h_N) + \\
& \frac{1}{2} L_f T(\mu+1) h_N
\end{aligned}$$

for sufficiently large $N (N \geqslant N_0)$ and all $t \in T$. Using the Bellman – Gronwall lemma [4], we obtain the inequality

$$\Delta(t) \leqslant [{}^1\!/_2 L_f T(\mu+1) h_N + L_f T \omega(h_N)] \exp[L_f T]$$

from which follows (5.2). It is clear that trajectories $x_N(t)$ are admissible in Problems A_N for any choice of sequence $\{\rho_N\}$, $N = 1, 2, \ldots$, from (5.2). By virtue of this we have

$$I_N^\circ - I_A^\circ \leqslant \varphi(x_N(t_1)) - \varphi(x^\circ(t_1)) \leqslant L_\varphi \Delta(t_1)$$

which ensures the upper bound in (5.3). To prove the left inequality in (5.3) we consider the sequence of pairs $\{x_N^\circ(t), u_N^\circ(t)\}$ optimal in Problems A_N and the trajectories $x^N(t)$ of continuous system (1.1), corresponding to the controls

$$u^N(t) = u_N^\circ(t^N), \quad t_0 \leqslant t \leqslant t_1, \quad N = 1, 2, \ldots \tag{5.4}$$

By virtue of (1.1), (1.5) and (5.4) we have

$$\Delta_1(t_1) \overset{\text{def}}{=} \| x_N^\circ(t_1) - x^N(t_1) \| \leqslant \int_{t_0}^{t_1} \| f(x_N^\circ(t^N), u_N^\circ(t^N), t^N)$$

$$- f(x^N(t), u_N^\circ(t^N), t) \| \, dt \leqslant L_f \int_{t_0}^{t_1} \Delta_1(t) \, dt + \frac{1}{2} L_f T (\mu + 1) h_N$$

From the Bellman — Gronwall lemma follows the inequality

$$\Delta_1(t_1) \leqslant {}^1/_2 L_f T (\mu + 1) h_N \exp [L_f T] \tag{5.5}$$

The lower bound in (5.3) now follows from (5.5) and the relations

$$I_N^\circ - I_A^\circ \geqslant \varphi(x^N(t_1)) - I_A^\circ - L_\varphi \Delta_1(t_1) \geqslant - L_\varphi \Delta_1(t_1)$$

Notes. 5.1. In the hypotheses of Theorem 5.1 there is the assumption that in Problem A exists an optimal control, continuous almost everywhere with a specifed modulus of continuity which occurs in bounds (5.2) and (5.3). Thus, the upper bound of the convergence rate directly in terms of the parameters of Problem A is connected with the effective conditions for the existence of optimal controls in specified classes of "accessible" functions. (See survey [9] for existence theorems of this type).

5.2. The methods developed in the present paper enable us to obtain analogous results on the convergence of discrete approximations for certain types of problems with mixed constraints on (x, u). In particular, if the control domain $U = U(x, t)$ depends continuously on both variables, then an analog of Theorem 4.1 is valid under an appropriate γ_N-approximation of set $U(x, t)$. If set $U(x, t)$ satisfies the Lipschitz condition, then by analogy with Theorem 5.1 we can estimate the quantities γ_N and the rate of the convergence $A_N \to A$ with respect to the functional.

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